Detailed Derivations for Paper “Blind Feedforward Cyclostationarity-Based Timing Estimation for Linear Modulations”

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1 Derivation of ζ_1 and ζ_2

First, some straightforward calculations lead to the following explicit relations for $G(k;\tau)$: $G(-1;\tau) = G(1;\tau)$,

$$G(1;\tau) = \begin{cases} \frac{\rho P}{8} & \text{if } \tau = 0 \ , \\ \frac{\rho P}{16} & \text{if } \tau = \pm\frac{P}{\rho} \ , \\ \frac{P^4 \sin\frac{\pi\tau\rho}{P}}{8\pi\tau(P^2-\tau^2\rho^2)} & \text{elsewhere} \ , \end{cases} \tag{1}$$

$$G(0;\tau) = \begin{cases} \frac{(4-\rho)P}{4} & \text{if } \tau = 0 \ , \\ \frac{2\rho P}{\pi}(\frac{\pi}{4}\sin\frac{\pi}{2\rho} - \frac{1}{3}\cos\frac{\pi}{2\rho}) & \text{if } \tau = \pm\frac{P}{2\rho} \ , \\ \frac{\rho P}{\pi}(\frac{1}{3}\sin\frac{\pi}{\rho} - \frac{\pi}{8}\cos\frac{\pi}{\rho}) & \text{if } \tau = \pm\frac{P}{\rho} \ , \\ \frac{2P}{\pi}\Big\{\frac{P}{2\tau}[\frac{5}{8}\sin\frac{(1-\rho)\pi\tau}{P} + \frac{3}{8}\sin\frac{(1+\rho)\pi\tau}{P}] + \frac{\tau\rho^2 P(\sin\frac{(1+\rho)\pi\tau}{P}+\sin\frac{(1-\rho)\pi\tau}{P})}{P^2-4\rho^2\tau^2} & \\ \qquad + \frac{1}{16}\frac{\tau\rho^2 P(\sin\frac{(1-\rho)\pi\tau}{P}-\sin\frac{(1+\rho)\pi\tau}{P})}{P^2-\rho^2\tau^2}\Big\} & \text{elsewhere} \ . \end{cases}$$

Since the pulse shape $h_c(t)$ is bandlimited in $[-(1+\rho)/2T, (1+\rho)/2T]$ with $0 < \rho \le 1$, under the assumption of $P \ge 3$, according to [3], the following expressions hold:

$$\begin{aligned} H(f) &= \frac{1}{T_s}H_c\left(\frac{f}{T_s}\right) = \frac{P}{T}H_c(F) \ , \\ H(f+k/P) &= \frac{1}{T_s}H_c\left(\frac{f+k/P}{T_s}\right) = \frac{P}{T}H_c\left(F+\frac{k}{T}\right) \ , \quad \text{for } |f| \le 1/P \text{ and } k = \pm 1 \ . \end{aligned}$$

Therefore, based on [4, Eqs. (6) and (7)], we can write:

$$\begin{aligned} \widetilde{\mathbf{\Gamma}}_{0,0} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{2x}(1;f)S_{2x}^*(-1;f-\frac{1}{P})\mathrm{d}f + \kappa P R_{2x}(1;0)R_{2x}(1;0) \\ &= \frac{1}{P^2}e^{-4i\pi\epsilon}\int_{-\frac{1}{2}}^{\frac{1}{2}} H^2(f)H^2(f-\frac{1}{P})\mathrm{d}f + \kappa P\frac{\rho^2}{64}e^{-4i\pi\epsilon} \\ &= \frac{P}{T^3}e^{-4i\pi\epsilon}\int_{-\frac{P}{2T}}^{\frac{P}{2T}} H_c^2(F)H_c^2(F-\frac{1}{T})\mathrm{d}F + \kappa P\frac{\rho^2}{64}e^{-4i\pi\epsilon} \\ &= \frac{P}{T^3}e^{-4i\pi\epsilon}\int_{-\frac{\rho}{2T}}^{\frac{\rho}{2T}} H_c^2(F+\frac{1}{2T})H_c^2(F-\frac{1}{2T})\mathrm{d}F + \kappa P\frac{\rho^2}{64}e^{-4i\pi\epsilon} = Pe^{-4i\pi\epsilon}\zeta_2 \ , \end{aligned}$$

where for the second equality we made use of [4, Eq. (2)] and (1), and

$$\zeta_2 := \frac{1}{T^3}\int_{-\frac{\rho}{2T}}^{\frac{\rho}{2T}} H_c^2(F+\frac{1}{2T})H_c^2(F-\frac{1}{2T})\mathrm{d}F + \frac{\kappa\rho^2}{64} \ .$$

Similarly, we can obtain $\mathbf{\Gamma}_{0,0} = P\zeta_1$, where:

$$\zeta_1 := \zeta_2 + \frac{2N_0}{T^2}\int_{-\frac{\rho}{2T}}^{\frac{\rho}{2T}} H_c^2(F+\frac{1}{2T})H_c(F-\frac{1}{2T})\mathrm{d}F + \frac{N_0^2}{T}\int_{-\frac{\rho}{2T}}^{\frac{\rho}{2T}} H_c(F+\frac{1}{2T})H_c(F-\frac{1}{2T})\mathrm{d}F \ .$$

2 Derivations of the Asymptotic Variance of $\hat{\epsilon}$ in [4, Eq. (15)] and $\hat{\alpha}_1$

The fourth-order time-varying correlation of $x(n)$ is defined as:

$$r_{4x}(n;\tau_1,\tau_2,\tau_3) := \mathrm{E}\{x^*(n)x^*(n+\tau_1)x(n+\tau_2)x(n+\tau_3)\} \ .$$

Consider only the case $\tau_1 = \tau_2 = \tau_3 = 0$, and based on [1, eq. 10.2.9], $r_{4x}(n;0,0,0)$ can be alternatively expressed as:

$$r_{4x}(n;0,0,0) = \mathrm{cum}(x^*(n),x^*(n),x(n),x(n)) + 2r_{2x}^2(n;0) = \kappa\sum_l h^4(n-lP-\epsilon P) + 2r_{2x}^2(n;0) \ .$$

It is not difficult to find that $r_{4x}(n;0,0,0)$ is also periodic with respect to n with period P, and its Fourier's coefficient at cycle $k=1$ is given by:

$$\begin{aligned} R_{4x}(1;0,0,0) &:= \frac{1}{P}\sum_{n=0}^{P-1} r_{4x}(n;0,0,0)e^{-2i\pi\frac{n}{P}} = \frac{\kappa}{P}\sum_n h^4(n-\epsilon P)e^{-2i\pi\frac{n}{P}} + 2R_{2x}(k;0)\otimes R_{2x}(k;0)|_{k=1} \\ &= \frac{\kappa}{P}\sum_n h^4(n-\epsilon P)e^{-2i\pi\frac{n}{P}} + 4R_{2x}(0;0)R_{2x}(1;0) \ , \end{aligned} \qquad (2)$$

where $\otimes$ stands for the circular convolution. Note that the following FT pairs hold:

$$h(n-\epsilon P) \longleftrightarrow H_{rc}(f)e^{-2i\pi f\epsilon P} \ , \quad h^2(n-\epsilon P) \longleftrightarrow e^{-2i\pi f\epsilon P}H_{rc}^{(2)}(f) \ , \quad H_{rc}^{(2)}(f) := H_{rc}(f) * H_{rc}(f) \ .$$

Based on Parseval's relation and [4, Eq. (2)], (2) can be expressed as:

$$\begin{aligned} R_{4x}(1;0,0,0) = \Big\{ &\frac{\kappa}{P}\int_{-1/2}^{1/2} H_{rc}^{(2)}(f)H_{rc}^{(2)}(f+\frac{1}{P})\mathrm{d}f \\ &+ \frac{4}{P}G(1;0)\Big(\frac{1}{P}G(0;0)+r_{2v}(0)\Big)\Big\} e^{-2i\pi\epsilon} = Q(1;0)e^{-2i\pi\epsilon} \ , \end{aligned}$$

where the factor contained within braces, denoted by $Q(1;0)$, is real-valued.

The OPT timing estimator takes the following form:

$$\hat{\epsilon} = -\frac{1}{2\pi}\arg\{\boldsymbol{\alpha}^{\mathrm{T}}\hat{\mathbf{R}}_x\} \ , \quad \boldsymbol{\alpha} := [1 \ \ \alpha_1]^{\mathrm{T}} \ , \quad \hat{\mathbf{R}}_x := [\hat{R}_{2x}(1;0) \ \ \hat{R}_{4x}(1;0,0,0)]^{\mathrm{T}}. \qquad (3)$$

Since the cyclic moment estimates $\hat{\mathbf{R}}_{2x}(1;0)$ and $\hat{\mathbf{R}}_{4x}(1;0,0,0)$ are asymptotically complex normal, so is any linear combination of them. By adopting the derivation presented in [4, Section III], we can obtain the following expression for the asymptotic variance of $\hat{\epsilon}$ in (3):

$$\mathrm{avar}(\hat{\epsilon}) = \frac{\boldsymbol{\alpha}^{\mathrm{T}}[\mathbf{\Pi} - \mathrm{re}(e^{4i\pi\epsilon}\widetilde{\mathbf{\Pi}})]\boldsymbol{\alpha}}{8\pi^2|\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\beta}|^2} \ , \qquad (4)$$

where:

$$\begin{aligned} \mathbf{\Pi} &:= \lim_{N\to\infty} N\mathrm{E}\{[\hat{\mathbf{R}}_x - \mathbf{R}_x][\hat{\mathbf{R}}_x - \mathbf{R}_x]^{\mathrm{H}}\} , \quad \widetilde{\mathbf{\Pi}} := \lim_{N\to\infty} N\mathrm{E}\{[\hat{\mathbf{R}}_x - \mathbf{R}_x][\hat{\mathbf{R}}_x - \mathbf{R}_x]^{\mathrm{T}}\} , \\ \mathbf{R}_x &:= [R_{2x}(1;0) \;\; R_{4x}(1;0,0,0)]^{\mathrm{T}} , \quad \boldsymbol{\beta} := \Big[\frac{G(1;0)}{P} \;\; Q(1;0)\Big]^{\mathrm{T}} . \end{aligned}$$

Hence, finding α_1 resorts to the standard Rayleigh quotient problem, whose solution is given by (c.f. [2, ch. 5]):

$$\hat{\alpha}_1^{(\mathrm{OPT})} = \frac{[0 \;\; 1] \cdot [\mathbf{\Pi} - \mathrm{re}(e^{4i\pi\epsilon}\widetilde{\mathbf{\Pi}})]^{-1}\boldsymbol{\beta}}{[1 \;\; 0] \cdot [\mathbf{\Pi} - \mathrm{re}(e^{4i\pi\epsilon}\widetilde{\mathbf{\Pi}})]^{-1}\boldsymbol{\beta}} . \tag{5}$$

Plugging (5) back into (4), we obtain:

$$\mathrm{avar}(\hat{\epsilon}^{(\mathrm{OPT})}) = \frac{1}{8\pi^2\boldsymbol{\beta}^{\mathrm{T}}[\mathbf{\Pi} - \mathrm{re}(e^{4i\pi\epsilon}\widetilde{\mathbf{\Pi}})]^{-1}\boldsymbol{\beta}} .$$

Now let us evaluate the entries of the asymptotic covariance matrices $\mathbf{\Pi}$ and $\widetilde{\mathbf{\Pi}}$. Obviously, the first entries of $\mathbf{\Pi}$ and $\widetilde{\mathbf{\Pi}}$ are given by $\mathbf{\Pi}_{0,0} = \mathbf{\Gamma}_{0,0}$, $\widetilde{\mathbf{\Pi}}_{0,0} = \widetilde{\mathbf{\Gamma}}_{0,0}$, respectively. Define the following mean-compensated stochastic processes:

$$e_2(n) := x^*(n)x(n) - r_{2x}(n;0) , \quad e_4(n) := x^*(n)x^*(n)x(n)x(n) - r_{4x}(n;0,0,0) ,$$

and let $r_{2e_2}(n;\tau) := \mathrm{E}\{e_2^*(n)e_2(n+\tau)\}$ and $R_{2e_2}(k;\tau)$ denote the time-varying and cyclic correlations of $e_2(n)$, respectively. In [3] , it was proved that $\mathbf{\Gamma}_{0,0} = S_{2e_2}(0;1/P)$, $\widetilde{\mathbf{\Gamma}}_{0,0} = S_{2e_2}(2;1/P)$, where $S_{2e_2}(k;f) := \sum_\tau R_{2e_2}(k;\tau)\exp\{-2i\pi\tau f\}$ is the cyclic spectrum of $e_2(n)$ at cycle k. The expressions of $\mathbf{\Gamma}$ and $\widetilde{\mathbf{\Gamma}}$ are established in [3], and given in [4, Eqs. (5) and (6)]. Here we present a new method which makes use of the circular convolution of cyclic correlations of $x(n)$ in an iterative way and can be applied to more general cases. Observe

$$\begin{aligned} r_{2e_2}(n;\tau) &= \mathrm{E}\{x^*(n)x(n)x^*(n+\tau)x(n+\tau)\} - r_{2x}(n;0)r_{2x}(n+\tau;0) \\ &= \mathrm{cum}(x^*(n), x(n), x^*(n+\tau), x(n+\tau)) + r_{2x}^2(n;\tau) \\ &= \kappa\sum_l h^2(n - lP - \epsilon P)h^2(n + \tau - lP - \epsilon P) + r_{2x}^2(n;\tau) , \end{aligned}$$

and

$$R_{2e_2}(k;\tau) = \frac{1}{P}\sum_{n=0}^{P-1} r_{2e_2}(n;\tau)e^{-2i\pi\frac{kn}{P}} = \frac{\kappa}{P}\sum_n h^2(n-\epsilon P)h^2(n+\tau-\epsilon P)e^{-2i\pi\frac{kn}{P}} + R_{2x}(k;\tau)\otimes R_{2x}(k;\tau).$$

Thus, we obtain:

$$S_{2e_2}(k;f) = \sum_\tau \Big(\frac{\kappa}{P}\sum_n h^2(n-\epsilon P)h^2(n+\tau-\epsilon P)e^{-2i\pi\frac{kn}{P}} + R_{2x}(k;\tau)\otimes R_{2x}(k;\tau)\Big)e^{-2i\pi\tau f} ,$$

which can be used for accurately evaluating the matrices $\mathbf{\Gamma}_{0,0}$ and $\widetilde{\mathbf{\Gamma}}_{0,0}$.

Similarly, the following expressions hold true:

$$\begin{aligned} \mathbf{\Pi}_{0,1} &= S_{e_4e_2}(0;1/P) , \quad \mathbf{\Pi}_{1,0} = S_{e_2e_4}(0;1/P) , \quad \mathbf{\Pi}_{1,1} = S_{2e_4}(0;1/P) , \\ \widetilde{\mathbf{\Pi}}_{0,1} &= S_{e_4e_2}(2;1/P) , \quad \widetilde{\mathbf{\Pi}}_{1,0} = S_{e_2e_4}(2;1/P) , \quad \widetilde{\mathbf{\Pi}}_{1,1} = S_{2e_4}(2;1/P) . \end{aligned}$$

The evaluation of the above expressions is similar to that of $\mathbf{\Gamma}_{0,0}$ and $\widetilde{\mathbf{\Gamma}}_{0,0}$, but involves the computation of higher-order (larger than second-order) cumulants and moments of $x(n)$, which is straightforward but exhibits too lengthy formulas, and therefore, will not be shown herein.

References

[1] B. Porat, *Digital Processing of Random Signals*, Prentice-Hall, 1994.

[2] P. Stoica and R. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997.

[3] Y. Wang, P. Ciblat, E. Serpedin and P. Loubaton, “Performance analysis of a class of non-data aided frequency offset and symbol timing estimators for flat-fading channels,” *IEEE Trans. on Signal Processing*, vol. 50, no. 9, pp. 2295-2305, Sept. 2002.

[4] Y. Wang, E. Serpedin and P. Ciblat, “Blind Feedforward Cyclostationarity-Based Timing Estimation for Linear Modulations,” *IEEE Trans. on Wireless Communications*, submitted, 2003.