Non-Data Aided Feedforward Carrier Frequency Offset Estimators for QAM Constellations: A Nonlinear Least-Squares Approach

Appendix 1: Derivation of Theorem 1

Considering the Taylor series expansion of $\sum_{k=0}^{P-1} \dot{\lambda}_k \exp(j\dot{\phi}_k + j2\pi(\dot{\alpha} + k/P)n)$ in the neighborhood of the true value $\boldsymbol{\theta} := [\lambda_0 \ \cdots \ \lambda_{P-1} \ \phi_0 \ \ldots \ \phi_{P-1} \ \alpha_0]^T$, we can write:

$$\sum_{k=0}^{P-1} \dot{\lambda}_k e^{j\dot{\phi}_k} e^{j2\pi(\dot{\alpha}+\frac{k}{P})n} = \sum_{k=0}^{P-1} \lambda_k e^{j\phi_k} e^{j2\pi(\alpha_0+\frac{k}{P})n} + \sum_{k=0}^{P-1} (\dot{\lambda}_k - \lambda_k) e^{j\phi_k} e^{j2\pi(\alpha_0+\frac{k}{P})n}$$
$$+ j\sum_{k=0}^{P-1} (\dot{\phi}_k - \phi_k)\lambda_k e^{j\phi_k} e^{j2\pi(\alpha_0+\frac{k}{P})n} + j2\pi n(\dot{\alpha} - \alpha_0)\sum_{k=0}^{P-1} \lambda_k e^{j\phi_k} e^{j2\pi(\alpha_0+\frac{k}{P})n} + \text{rem},$$

where rem is the high-order remainder term which can be neglected. Then we can approximate (16) by:

$$J(\dot{\boldsymbol{\theta}}) \doteq \frac{1}{2N} \sum_{n=0}^{N-1} \Big| x^4(n) - \sum_{k=0}^{P-1} \lambda_k e^{j\phi_k} e^{j2\pi(\alpha_0+\frac{k}{P})n} - \sum_{k=0}^{P-1} (\dot{\lambda}_k - \lambda_k) e^{j\phi_k} e^{j2\pi(\alpha_0+\frac{k}{P})n}$$
$$- j\sum_{k=0}^{P-1} (\dot{\phi}_k - \phi_k)\lambda_k e^{j\phi_k} e^{j2\pi(\alpha_0+\frac{k}{P})n} - j2\pi n(\dot{\alpha} - \alpha_0)\sum_{k=0}^{P-1} \lambda_k e^{j\phi_k} e^{j2\pi(\alpha_0+\frac{k}{P})n} \Big|^2 . \quad (1)$$

Setting $\partial J(\hat{\boldsymbol{\theta}})/\partial\dot{\lambda}_k = 0$ for $k = 0, \ldots, \ P-1$, we obtain:

$$\hat{\lambda}_k = \text{re}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} x^4(n) e^{-j\phi_k} e^{-j2\pi(\alpha_0+\frac{k}{P})n} \right\} - \sum_{\substack{l=0\\l\neq k}}^{P-1} \hat{\lambda}_l \text{re}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} e^{j(\phi_k-\phi_l)} e^{j2\pi\frac{k-l}{P}n} \right\}$$
$$+ \sum_{\substack{l=0\\l\neq k}}^{P-1} \hat{\lambda}_l (\hat{\phi}_l - \phi_l) \text{im}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} e^{j(\phi_k-\phi_l)} e^{j2\pi\frac{k-l}{P}n} \right\}$$
$$+ 2\pi N(\hat{\alpha} - \alpha_0) \sum_{\substack{l=0\\l\neq k}}^{P-1} \hat{\lambda}_l \text{im}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} \frac{n}{N} e^{j(\phi_k-\phi_l)} e^{j2\pi\frac{k-l}{P}n} \right\} . \quad (2)$$

To compute the individual factors in the R.H.S. of (2), the following well-known result will be used extensively [2]:

Lemma 1. *With k denoting a positive integer and $\delta(\omega)$ standing for a Kronecker delta, it holds that:*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}(\frac{n}{N})^k e^{j(\omega n+\phi)} = \frac{e^{j\phi}\delta(\omega)}{k+1} \ . \tag{3}$$

Using (3), we can further approximate (2) by:

$$\hat{\lambda}_k = \frac{1}{N\lambda_k}\mathrm{re}\left\{\sum_{n=0}^{N-1} x^4(n)\lambda_k e^{-j\phi_k} e^{-j2\pi(\alpha_0+\frac{k}{P})n}\right\} \ . \tag{4}$$

Following the same procedure, i.e., by setting:

$$\frac{\partial J(\hat{\boldsymbol{\theta}})}{\partial\dot{\phi}_k} = 0 \ , \ k=0,\ldots,\ P-1, \qquad \frac{\partial J(\hat{\boldsymbol{\theta}})}{\partial\dot{\alpha}} = 0 \ ,$$

and using Lemma 1, the following expressions can be obtained:

$$\hat{\phi}_k - \phi_k = \frac{1}{N\lambda_k}\mathrm{im}\left\{\sum_{n=0}^{N-1} x^4(n) e^{-j\phi_k} e^{-j2\pi(\alpha_0+\frac{k}{P})n}\right\} - \pi N(\hat{\alpha}-\alpha_0) \ , \tag{5}$$

$$\begin{aligned} N(\hat{\alpha}-\alpha_0) = &\frac{3}{2\pi N\Lambda^2}\mathrm{im}\left\{\sum_{n=0}^{N-1}\frac{n}{N}x^4(n)\sum_{l=0}^{P-1}\lambda_l e^{-j\phi_l}e^{-j2\pi(\alpha_0+\frac{l}{P})n}\right\} \\ &-\frac{3}{4\pi\Lambda^2}\sum_{l=0}^{P-1}\lambda_l^2(\hat{\phi}_l-\phi_l) \ , \end{aligned} \tag{6}$$

where $\Lambda^2 := \sum_{k=0}^{P-1}\lambda_k^2$. Solving (5) and (6), we can express $\hat{\phi}_k,\ k=0,\ldots,P-1$ and $\hat{\alpha}$ in terms of the true value $\boldsymbol{\theta}$ and $x^4(n)$ by:

$$\begin{aligned} \hat{\phi}_k - \phi_k = &\frac{1}{N\lambda_k^2}\mathrm{im}\left\{\sum_{n=0}^{N-1} x^4(n)\lambda_k e^{-j\phi_k} e^{-j2\pi(\alpha_0+\frac{k}{P})n}\right\} \\ &+\frac{3}{N\Lambda^2}\mathrm{im}\left\{\sum_{n=0}^{N-1} x^4(n)\sum_{l=0}^{P-1}\lambda_l e^{-j\phi_l}e^{-j2\pi(\alpha_0+\frac{l}{P})n}\right\} \\ &-\frac{6}{N\Lambda^2}\mathrm{im}\left\{\sum_{n=0}^{N-1}\frac{n}{N}x^4(n)\sum_{l=0}^{P-1}\lambda_l e^{-j\phi_l}e^{-j2\pi(\alpha_0+\frac{l}{P})n}\right\} \ , \end{aligned} \tag{7}$$

$$\begin{aligned} N(\hat{\alpha}-\alpha_0) = &\frac{6}{\pi N\Lambda^2}\mathrm{im}\left\{\sum_{n=0}^{N-1}\frac{n}{N}x^4(n)\sum_{l=0}^{P-1}\lambda_l e^{-j\phi_l}e^{-j2\pi(\alpha_0+\frac{l}{P})n}\right\} \\ &-\frac{3}{\pi N\Lambda^2}\mathrm{im}\left\{\sum_{n=0}^{N-1} x^4(n)\sum_{l=0}^{P-1}\lambda_l e^{-j\phi_l}e^{-j2\pi(\alpha_0+\frac{l}{P})n}\right\} \ . \end{aligned} \tag{8}$$

Next, let us write the above expressions in matrix form*:

$$
\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} = \boldsymbol{m} + \boldsymbol{H}\boldsymbol{b} \; , \; \boldsymbol{m} := [-\lambda_0 \; \cdots \; -\lambda_{P-1} \; 0 \; \cdots \; 0]^T \; ,
$$

$$
\boldsymbol{H} := \begin{bmatrix}
\frac{1}{\lambda_0} & 0 & 0 & \dots & 0 & 0 & \dots & \dots & \dots & 0 & 0 \\
0 & \frac{1}{\lambda_1} & 0 & \dots & 0 & 0 & \dots & \dots & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & \frac{1}{\lambda_{P-1}} & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \frac{1}{\lambda_0^2} + \frac{3}{\Lambda^2} & \frac{3}{\Lambda^2} & \dots & \dots & \frac{3}{\Lambda^2} & -\frac{6}{\Lambda^2} \\
0 & 0 & \dots & 0 & 0 & \frac{3}{\Lambda^2} & \frac{1}{\lambda_1^2} + \frac{3}{\Lambda^2} & \frac{3}{\Lambda^2} & \dots & \frac{3}{\Lambda^2} & -\frac{6}{\Lambda^2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & \frac{3}{\Lambda^2} & \frac{3}{\Lambda^2} & \dots & \frac{3}{\Lambda^2} & \frac{1}{\lambda_{P-1}^2} + \frac{3}{\Lambda^2} & -\frac{6}{\Lambda^2} \\
0 & 0 & \dots & 0 & 0 & -\frac{3}{\pi\Lambda^2} & -\frac{3}{\pi\Lambda^2} & \dots & -\frac{3}{\pi\Lambda^2} & -\frac{3}{\pi\Lambda^2} & \frac{6}{\pi\Lambda^2}
\end{bmatrix} ,
$$

$$
\boldsymbol{b} := \begin{bmatrix}
\frac{1}{N}\mathrm{re}\left\{ \sum_{n=0}^{N-1} x^4(n) \lambda_0 e^{-j\phi_0} e^{-j2\pi\alpha_0 n} \right\} \\
\frac{1}{N}\mathrm{re}\left\{ \sum_{n=0}^{N-1} x^4(n) \lambda_1 e^{-j\phi_1} e^{-j2\pi(\alpha_0 + \frac{1}{P})n} \right\} \\
\vdots \\
\frac{1}{N}\mathrm{re}\left\{ \sum_{n=0}^{N-1} x^4(n) \lambda_{P-1} e^{-j\phi_{P-1}} e^{-j2\pi(\alpha_0 + \frac{P-1}{P})n} \right\} \\
\frac{1}{N}\mathrm{im}\left\{ \sum_{n=0}^{N-1} x^4(n) \lambda_0 e^{-j\phi_0} e^{-j2\pi\alpha_0 n} \right\} \\
\frac{1}{N}\mathrm{im}\left\{ \sum_{n=0}^{N-1} x^4(n) \lambda_1 e^{-j\phi_1} e^{-j2\pi(\alpha_0 + \frac{1}{P})n} \right\} \\
\vdots \\
\frac{1}{N}\mathrm{im}\left\{ \sum_{n=0}^{N-1} x^4(n) \lambda_{P-1} e^{-j\phi_{P-1}} e^{-j2\pi(\alpha_0 + \frac{P-1}{P})n} \right\} \\
\frac{1}{N}\mathrm{im}\left\{ \sum_{n=0}^{N-1} \frac{n}{N} x^4(n) \sum_{l=0}^{P-1} e^{-j\phi_l} e^{-j2\pi(\alpha_0 + \frac{l}{P})n} \right\}
\end{bmatrix} . \tag{9}
$$

Using (12) and Lemma 1, it is straightforward to verify that $\lim_{N\to\infty} \mathrm{E}\{\boldsymbol{b}\} = [\lambda_0^2 \; \lambda_1^2 \dots \; \lambda_{P-1}^2 \; 0 \; \dots \; 0]^T$, hence the asymptotic unbiasedness of $\hat{\boldsymbol{\theta}}$ follows, i.e., $\lim_{N\to\infty} \mathrm{E}\{\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\} = \mathbf{0}$.

Since in (9) only $\boldsymbol{b}$ is random, the asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}$ can be simplified as:

$$
\boldsymbol{\Sigma} := \lim_{N\to\infty} N\mathrm{cov}(\hat{\boldsymbol{\theta}}) = \boldsymbol{H} \lim_{N\to\infty} [N\mathrm{cov}(\boldsymbol{b})]\boldsymbol{H}^T := \boldsymbol{H}\boldsymbol{B}\boldsymbol{H}^T \; . \tag{10}
$$

* In the following we replace $\hat{\alpha}$ and α_0 by $N\hat{\alpha}$ and $N\alpha_0$ in $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$, respectively.

There are $(2P+1)\times(2P+1)$ entries $\mathbf{\Sigma}_{k,l},\ k,l\in[0,\ 2P]$, but we are only interested in $\gamma=\mathbf{\Sigma}_{2P,2P}$. Due to the special structure of $\mathbf{H}$, it is not difficult to find that:

$$\gamma=\mathbf{u}^{\mathrm{T}}\mathbf{B}^{(s)}\mathbf{u}\ ,\ \ \mathbf{u}:=[-\frac{3}{\pi\Lambda^2}\ \ -\frac{3}{\pi\Lambda^2}\ \ \cdots-\frac{3}{\pi\Lambda^2}\ \ \frac{6}{\pi\Lambda^2}\ ]^{\mathrm{T}}\ , \tag{11}$$

$$\mathbf{B}^{(s)}:=\lim_{N\to\infty}[N\mathrm{cov}(\boldsymbol{b}_s)]\ ,$$

$$\boldsymbol{b}_s:=\begin{bmatrix}\frac{1}{N}\mathrm{im}\left\{\sum_{n=0}^{N-1}x^4(n)\lambda_0e^{-j\phi_0}e^{-j2\pi\alpha_0 n}\right\}\\ \frac{1}{N}\mathrm{im}\left\{\sum_{n=0}^{N-1}x^4(n)\lambda_1e^{-j\phi_1}e^{-j2\pi(\alpha_0+\frac{1}{P})n}\right\}\\ \vdots\\ \frac{1}{N}\mathrm{im}\left\{\sum_{n=0}^{N-1}x^4(n)\lambda_{P-1}e^{-j\phi_{P-1}}e^{-j2\pi(\alpha_0+\frac{P-1}{P})n}\right\}\\ \frac{1}{N}\mathrm{im}\left\{\sum_{n=0}^{N-1}\frac{n}{N}x^4(n)\sum_{l=0}^{P-1}e^{-j\phi_l}e^{-j2\pi(\alpha_0+\frac{l}{P})n}\right\}\end{bmatrix}\ .$$

The entries $\mathbf{B}^{(s)}_{l_1,l_2},\ l_1,l_2\in[0,P-1]$ of matrix $\mathbf{B}^{(s)}$ can be expressed as:

$$\begin{aligned}\boldsymbol{B}^{(s)}_{l_1,l_2}=&\lim_{N\to\infty}N\mathrm{cov}\Big(\frac{1}{N}\mathrm{im}\Big\{\sum_{n=0}^{N-1}x^4(n)\lambda_{l_1}e^{-j\phi_{l_1}}e^{-j2\pi(\alpha_0+\frac{l_1}{P})n}\Big\},\\ &\qquad\frac{1}{N}\mathrm{im}\Big\{\sum_{n=0}^{N-1}x^4(n)\lambda_{l_2}e^{-j\phi_{l_2}}e^{-j2\pi(\alpha_0+\frac{l_2}{P})n}\Big\}\Big)\\ =&\lim_{N\to\infty}\frac{1}{2N}\Big\{\mathrm{re}\Big\{\mathrm{cov}\Big(\sum_{n=0}^{N-1}x^4(n)\lambda_{l_1}e^{-j\phi_{l_1}}e^{-j2\pi(\alpha_0+\frac{l_1}{P})n},\\ &\qquad\sum_{n=0}^{N-1}x^{*4}(n)\lambda_{l_2}e^{j\phi_{l_2}}e^{j2\pi(\alpha_0+\frac{l_2}{P})n}\Big)\Big\}\\ &-\mathrm{re}\Big\{\mathrm{cov}\Big(\sum_{n=0}^{N-1}x^4(n)\lambda_{l_1}e^{-j\phi_{l_1}}e^{-j2\pi(\alpha_0+\frac{l_1}{P})n},\\ &\qquad\sum_{n=0}^{N-1}x^4(n)\lambda_{l_2}e^{-j\phi_{l_2}}e^{-j2\pi(\alpha_0+\frac{l_2}{P})n}\Big)\Big\}\Big\}\ .\end{aligned} \tag{12}$$

From (13), we can obtain the following time-varying covariances:

$$\mathrm{cov}\{x^4(n_1),\ x^4(n_2)\}=\mathrm{E}\{e(n_1)e(n_2)\}=\widetilde{c}_{2e}(n_2;n_1-n_2)\ , \tag{13}$$

$$\mathrm{cov}\{x^4(n_1),\ x^{*4}(n_2)\}=\mathrm{E}\{e(n_1)e^*(n_2)\}=c_{2e}(n_2;n_1-n_2)\ . \tag{14}$$

Since $v(n)$ satisfies the mixing condition (**AS3**), $w(n)$ has finite moments and $h(n)$ has finite memory, it follows that $e(n)$ (defined in (13)) also has finite moments, i.e., $\widetilde{c}_{2e}(n;\tau)<\infty$ and $c_{2e}(n;\tau)<\infty$. Substituting (13) and (14) into (12), we can express the first term of the R.H.S. of (12) as follows:

$$\lim_{N\to\infty}\frac{1}{2N}\Big\{\mathrm{re}\Big\{\mathrm{cov}\Big(\sum_{n=0}^{N-1}x^4(n)\lambda_{l_1}e^{-j\phi_{l_1}}e^{-j2\pi(\alpha_0+\frac{l_1}{P})n},$$

$$\sum_{n=0}^{N-1}x^{*4}(n)\lambda_{l_2}e^{j\phi_{l_2}}e^{j2\pi(\alpha_0+\frac{l_2}{P})n}\Big)\Big\}$$

$$=\mathrm{re}\Big\{\lim_{N\to\infty}\frac{1}{2N}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\lambda_{l_1}\lambda_{l_2}e^{-j(\phi_{l_1}-\phi_{l_2})}e^{-j2\pi[\alpha_0(n_1-n_2)+\frac{l_1n_1-l_2n_2}{P}]}c_{2e}(n_2;n_1-n_2)\Big\}$$

$$=\mathrm{re}\Big\{\lim_{N\to\infty}\frac{1}{2N}\sum_{\tau=-(N-1)}^{N-1}\sum_{n=0}^{N-1-|\tau|}\lambda_{l_1}\lambda_{l_2}e^{-j(\phi_{l_1}-\phi_{l_2})}e^{-j2\pi(\alpha_0+\frac{l_1}{P})\tau}e^{-j2\pi\frac{l_1-l_2}{P}n}c_{2e}(n;\tau)\Big\}$$

$$=\frac{1}{2}\mathrm{re}\Big\{\lambda_{l_1}\lambda_{l_2}e^{-j(\phi_{l_1}-\phi_{l_2})}\lim_{N\to\infty}\sum_{\tau=-(N-1)}^{N-1}\Big[\frac{1}{N}\sum_{n=0}^{N-1-|\tau|}c_{2e}(n;\tau)e^{-j2\pi\frac{l_1-l_2}{P}n}\Big]e^{-j2\pi(\alpha_0+\frac{l_1}{P})\tau}\Big\}$$

$$=\frac{1}{2}\mathrm{re}\Big\{\lambda_{l_1}\lambda_{l_2}e^{-j(\phi_{l_1}-\phi_{l_2})}\lim_{N\to\infty}\sum_{\tau=-(N-1)}^{N-1}C_{2e}\Big(\frac{l_1-l_2}{P};\tau\Big)e^{-j2\pi(\alpha_0+\frac{l_1}{P})\tau}\Big\}$$

$$=\frac{1}{2}\mathrm{re}\Big\{\lambda_{l_1}\lambda_{l_2}e^{-j(\phi_{l_1}-\phi_{l_2})}S_{2e}\Big(\frac{l_1-l_2}{P};\alpha_0+\frac{l_1}{P}\Big)\Big\}$$

$$=\frac{1}{2}\mathrm{re}\Big\{\widetilde{C}_{4x}^{*}\Big(\alpha_0+\frac{l_1}{P};\mathbf{0}\Big)\widetilde{C}_{4x}\Big(\alpha_0+\frac{l_2}{P};\mathbf{0}\Big)S_{2e}\Big(\frac{l_1-l_2}{P};\alpha_0+\frac{l_1}{P}\Big)\Big\}\ ,$$

where we have replaced the double sum over n_1 and n_2 by the double sum over $n := n_2$ and $\tau := n_1 - n_2$, and used the Lemma 1. Also, $C_{2e}(\alpha;\tau)$ stands for the unconjugate cyclic correlation of $e(n)$ [1]. Similarly, the second term of the R.H.S. of (12) can be expressed as:

$$\lim_{N\to\infty}\frac{1}{2N}\Big\{\mathrm{re}\Big\{\mathrm{cov}\Big(\sum_{n=0}^{N-1}x^4(n)\lambda_{l_1}e^{-j\phi_{l_1}}e^{-j2\pi(\alpha_0+\frac{l_1}{P}n)},$$

$$\sum_{n=0}^{N-1}x^{4}(n)\lambda_{l_2}e^{-j\phi_{l_2}}e^{-j2\pi(\alpha_0+\frac{l_2}{P})n}\Big)\Big\}$$

$$=\frac{1}{2}\mathrm{re}\Big\{\widetilde{C}_{4x}^{*}\Big(\alpha_0+\frac{l_1}{P};\mathbf{0}\Big)\widetilde{C}_{4x}^{*}\Big(\alpha_0+\frac{l_2}{P};\mathbf{0}\Big)\widetilde{S}_{2e}\Big(2\alpha_0+\frac{l_1+l_2}{P};\alpha_0+\frac{l_1}{P}\Big)\Big\}\ .$$

Therefore, for $l_1, l_2 \in [0, P-1]$, we obtain:

$$\mathbf{B}_{l_1,l_2}^{(s)}=\frac{1}{2}\mathrm{re}\Big\{\widetilde{C}_{4x}^{*}\Big(\alpha_0+\frac{l_1}{P};\mathbf{0}\Big)\widetilde{C}_{4x}\Big(\alpha_0+\frac{l_2}{P};\mathbf{0}\Big)S_{2e}\Big(\frac{l_1-l_2}{P};\alpha_0+\frac{l_1}{P}\Big)\Big\}$$
$$-\frac{1}{2}\mathrm{re}\Big\{\widetilde{C}_{4x}^{*}\Big(\alpha_0+\frac{l_1}{P};\mathbf{0}\Big)\widetilde{C}_{4x}^{*}\Big(\alpha_0+\frac{l_2}{P};\mathbf{0}\Big)\widetilde{S}_{2e}\Big(2\alpha_0+\frac{l_1+l_2}{P};\alpha_0+\frac{l_1}{P}\Big)\Big\}\ .$$

Using similar arguments, the following expression can be derived for $l \in [0, P-1]$:

$$\mathbf{B}^{(s)}_{P,l} = \mathbf{B}^{(s)}_{l,P} = \frac{1}{4}\sum_{k=0}^{P-1}\Big\{\mathrm{re}\Big\{\widetilde{C}^*_{4x}\Big(\alpha_0+\frac{l}{P};\mathbf{0}\Big)\widetilde{C}_{4x}\Big(\alpha_0+\frac{k}{P};\mathbf{0}\Big)S_{2e}\Big(\frac{l-k}{P};\alpha_0+\frac{l}{P}\Big)\Big\}$$
$$-\frac{1}{4}\sum_{k=0}^{P-1}\mathrm{re}\Big\{\widetilde{C}^*_{4x}\Big(\alpha_0+\frac{l}{P};\mathbf{0}\Big)\widetilde{C}^*_{4x}\Big(\alpha_0+\frac{k}{P};\mathbf{0}\Big)\widetilde{S}_{2e}\Big(2\alpha_0+\frac{l+k}{P};\alpha_0+\frac{l}{P}\Big)\Big\}\Big\}\ ,$$

and

$$\mathbf{B}^{(s)}_{P,P} = \frac{1}{6}\sum_{k,l=0}^{P-1}\Big\{\mathrm{re}\Big\{\widetilde{C}^*_{4x}\Big(\alpha_0+\frac{l}{P};\mathbf{0}\Big)\widetilde{C}_{4x}\Big(\alpha_0+\frac{k}{P};\mathbf{0}\Big)S_{2e}\Big(\frac{l-k}{P};\alpha_0+\frac{l}{P}\Big)\Big\}$$
$$-\frac{1}{6}\sum_{k,l=0}^{P-1}\mathrm{re}\Big\{\widetilde{C}^*_{4x}\Big(\alpha_0+\frac{l}{P};\mathbf{0}\Big)\widetilde{C}^*_{4x}\Big(\alpha_0+\frac{k}{P};\mathbf{0}\Big)\widetilde{S}_{2e}\Big(2\alpha_0+\frac{l+k}{P};\alpha_0+\frac{l}{P}\Big)\Big\}\Big\}\ .$$

Based on (11), after some lengthy calculations, we express γ as:

$$\gamma = \frac{3}{2\pi^2\Lambda^4}\sum_{l_1,l_2=0}^{P-1}\Big\{\mathrm{re}\Big\{\widetilde{C}^*_{4x}\Big(\alpha_0+\frac{l_1}{P};\mathbf{0}\Big)\widetilde{C}_{4x}\Big(\alpha_0+\frac{l_2}{P};\mathbf{0}\Big)S_{2e}\Big(\frac{l_1-l_2}{P};\alpha_0+\frac{l_1}{P}\Big)\Big\}$$
$$-\mathrm{re}\Big\{\widetilde{C}^*_{4x}\Big(\alpha_0+\frac{l_1}{P};\mathbf{0}\Big)\widetilde{C}^*_{4x}\Big(\alpha_0+\frac{l_2}{P};\mathbf{0}\Big)\widetilde{S}_{2e}\Big(2\alpha_0+\frac{l_1+l_2}{P};\alpha_0+\frac{l_1}{P}\Big)\Big\}\Big\}\ ,$$

and when the above expression is rewritten in matrix form, the equation (17) follows.

Appendix 2: Derivation of Propositions 1 and 2 (sketch)

Our purpose is to evaluate the unconjugate/conjugate cyclic spectra $S_{2e}(\alpha; f)$ and $\widetilde{S}_{2e}(\alpha; f)$ corresponding to two oversampling factors $P = 1$ and $P > 1$, respectively.

According to the definition of the additive noise $e(n)$ (13), we can express its unconjugate/conjugate time-varying correlations as:

$$\begin{aligned}
&c_{2e}(n;\tau) := \mathrm{E}\{e^*(n)e(n+\tau)\}\\
&= \mathrm{E}\Big\{[x^4(n)-\tilde{c}_{4x}(n;\mathbf{0})]^*[x^4(n+\tau)-\tilde{c}_{4x}(n+\tau;\mathbf{0})]\Big\}\\
&= 16m_{2x}(n;\tau)m_{6x}(n;0,0,\tau,\tau,\tau)+18m^2_{4x}(n;0,\tau,\tau)\\
&-144m^2_{2x}(n;\tau)m_{4x}(n;0,\tau,\tau)+144m^4_{2x}(n;\tau)\\
&+\mathrm{cum}\{\underbrace{x^*(n),\cdots x^*(n)}_{4},\underbrace{x(n+\tau),\cdots x(n+\tau)}_{4}\}\ , \qquad (15)\\
&\tilde{c}_{2e}(n;\tau) := \mathrm{E}\{e(n)e(n+\tau)\}\\
&= \mathrm{E}\Big\{[x^4(n)-\tilde{c}_{4x}(n;\mathbf{0})][x^4(n+\tau)-\tilde{c}_{4x}(n+\tau;\mathbf{0})]\Big\}
\end{aligned}$$

$$= 16\mathrm{E}\{x(n)x^3(n+\tau)\}\mathrm{E}\{x^3(n)x(n+\tau)\} + 18\mathrm{E}^2\{x^2(n)x^2(n+\tau)\}$$
$$+\mathrm{cum}\{\underbrace{x(n),\cdots x(n)}_{4},\underbrace{x(n+\tau),\cdots x(n+\tau)}_{4}\}\ , \qquad (16)$$

respectively.

Case 1. Evaluation of $S_{2e}(\alpha_0)$ for $P=1$

Note that if $P=1$, the moments m_{lx} are independent of the time index n,

$$\mathrm{cum}\{\underbrace{x^*(n),\cdots x^*(n)}_{4},\underbrace{x(n+\tau),\cdots x(n+\tau)}_{4}\} = \kappa_8 e^{j2\pi\alpha_0\tau}\sum_l h^{*4}(l)h^4(l+\tau)$$

and $S_{2e}(\alpha_0)$ in Proposition 1 can be obtained by plugging the above expression into (15) and taking the Fourier transform of the sequence $\{c_{2e}(\tau)\}_\tau$.

Case 2. Evaluation of $\widetilde{S}_{2e}(2\alpha_0;\alpha_0)$ for $P=1$

The following expressions can be derived due to the circularity of the transmitted signal $w(n)$:

$$\mathrm{E}\{x(n)x^3(n+\tau)\} = \mathrm{cum}\{x(n),x(n+\tau),x(n+\tau),x(n+\tau)\}$$
$$= \tilde{\kappa}_4 e^{j2\pi f_e(4n+3\tau)}\sum_l h(l)h^3(l+\tau)\ ,$$
$$\mathrm{E}\{x^3(n)x(n+\tau)\} = \mathrm{cum}\{x(n),x(n),x(n),x(n+\tau)\}$$
$$= \tilde{\kappa}_4 e^{j2\pi f_e(4n+\tau)}\sum_l h^3(l)h(l+\tau)\ ,$$
$$\mathrm{E}\{x^2(n)x^2(n+\tau)\} = \mathrm{cum}\{x(n),x(n),x(n+\tau),x(n+\tau)\}$$
$$= \tilde{\kappa}_4 e^{j2\pi f_e(4n+2\tau)}\sum_l h^2(l)h^2(l+\tau)\ ,$$
$$\mathrm{cum}\{\underbrace{x(n),\cdots x(n)}_{4},\underbrace{x(n+\tau),\cdots x(n+\tau)}_{4}\} = \tilde{\kappa}_8 e^{j2\pi f_e(8n+4\tau)}\sum_l h^4(l)h^4(l+\tau).$$

Then, the conjugate cyclic correlation $\widetilde{C}_{2e}(2\alpha_0;\tau)$ can be obtained as:

$$\widetilde{C}_{2e}(2\alpha_0;\tau) := \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\tilde{c}_{2e}(n;\tau)e^{-j4\pi\alpha_0 n}$$
$$= e^{j2\pi\alpha_0\tau}\Big[\tilde{\kappa}_8\sum_l h^4(l)h^4(l+\tau) + 16\tilde{\kappa}_4^2\sum_l h(l)h^3(l+\tau)\cdot\sum_l h^3(l)h(l+\tau)$$
$$+18\tilde{\kappa}_4^2\Big(\sum_l h^2(l)h^2(l+\tau)\Big)^2\Big]\ .$$

Finally, the expression of $\widetilde{S}_{2e}(2\alpha_0;\alpha_0)$ follows by taking the Fourier transform of the sequence $\{\widetilde{C}_{2e}(2\alpha_0;\tau)\}_\tau$ at the frequency α_0.

Case 3. Evaluation of $S_{2e}(k/P;\alpha_0+l/P)$ for $P>1$

When $P>1$, the last term of (15) can be expressed as:

$$\begin{aligned}&\text{cum}\{\underbrace{x^*(n),\cdots x^*(n)}_{4},\underbrace{x(n+\tau),\cdots x(n+\tau)}_{4}\}\\ &=\kappa_8 e^{j2\pi\alpha_0\tau}\sum_l h^{*4}(n-lP)h^4(n+\tau-lP)\ .\end{aligned}$$

The cyclic correlation coefficient at cycle k/P and the cyclic spectrum at frequency α_0+l/P of $e(n)$ can be expressed as:

$$\begin{aligned}C_{2e}\Big(\frac{k}{P};\tau\Big)&=\frac{1}{P}\sum_{n=0}^{P-1}c_{2e}(n;\tau)e^{-j2\pi\frac{kn}{P}}\ ,\\ S_{2e}\Big(\frac{k}{P};\alpha_0+\frac{l}{P}\Big)&=\sum_\tau C_{2e}\Big(\frac{k}{P};\tau\Big)e^{-j2\pi(\alpha_0+\frac{l}{P})\tau}\ .\end{aligned}$$

Since $m_{lx}(n;\boldsymbol{\tau})=\sum_{k=0}^{P-1}M_{lx}(k;\boldsymbol{\tau})\exp(j2\pi kn/P)$ for $l=4,6$, we obtain:

$$\begin{aligned}C_{2e}\Big(\frac{k}{P};\tau\Big)=&\,16\mathcal{V}_1+18\mathcal{V}_2-144\mathcal{V}_3+144\mathcal{V}_4\\ &+\frac{\kappa_8}{P}e^{j2\pi\alpha_0\tau}\sum_n h^{*4}(n)h^4(n+\tau)e^{-j2\pi\frac{kn}{P}}\ ,\end{aligned}$$

where $\mathcal{V}_i,\ i=1,\ldots,4$, are defined as in Proposition 2. Hence, we obtain:

$$\begin{aligned}&S_{2e}\Big(\frac{k}{P};\alpha_0+\frac{l}{P}\Big)=\sum_\tau(16\mathcal{V}_1+18\mathcal{V}_2-144\mathcal{V}_3+144\mathcal{V}_4)e^{-j2\pi(\alpha_0+\frac{l}{P})\tau}\\ &+\sum_\tau\frac{\kappa_8}{P}e^{j2\pi\alpha_0\tau}\sum_n h^{*4}(n)h^4(n+\tau)e^{-j2\pi\frac{kn}{P}}e^{-j2\pi(\alpha_0+\frac{l}{P})\tau}\ .\end{aligned}\tag{17}$$

Note that the last term of (17) can be expressed as:

$$\begin{aligned}&\sum_\tau\frac{\kappa_8}{P}e^{j2\pi\alpha_0\tau}\sum_n h^{*4}(n)h^4(n+\tau)e^{-j2\pi\frac{kn}{P}}e^{-j2\pi(\alpha_0+\frac{l}{P})\tau}\\ &=\frac{\kappa_8}{P}\sum_\tau\sum_n h^{*4}(n)h^4(n+\tau)e^{-j2\pi\frac{kn}{P}}e^{-j2\pi\frac{l}{P}\tau}\\ &=\frac{\kappa_8}{P}\sum_{\tau_1}h^4(\tau_1)e^{-j2\pi\frac{l}{P}\tau_1}\sum_n h^{*4}(n)e^{-j2\pi\frac{(k-l)n}{P}}\\ &=\frac{\kappa_8 P}{\widetilde{\kappa}_4^2}\widetilde{C}_{4x}\Big(\alpha_0+\frac{l}{P};\mathbf{0}\Big)\widetilde{C}^*_{4x}\Big(\alpha_0+\frac{l-k}{P};\mathbf{0}\Big)\ ,\end{aligned}$$

then, $S_{2e}(k/P;\alpha_0+l/P)$ in Proportion 2 is obtained.

Case 4. Evaluation of $\widetilde{S}_{2e}(2\alpha_0+k/P;\alpha_0+l/P)$ *for* $P>1$

Following the similar procedure presented in Case 2, it is not difficult to show:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{E}\{x(n)x^3(n+\tau)\}e^{-j2\pi(\alpha_0+\frac{k}{P})n}=e^{j6\pi f_e\tau}\widetilde{C}_{4x_1}(k;\tau)\ ,$$
$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{E}\{x^2(n)x^2(n+\tau)\}e^{-j2\pi(\alpha_0+\frac{k}{P})n}=e^{j4\pi f_e\tau}\widetilde{C}_{4x_2}(k;\tau)\ ,$$
$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{E}\{x^3(n)x(n+\tau)\}e^{-j2\pi(\alpha_0+\frac{k}{P})n}=e^{j2\pi f_e\tau}\widetilde{C}_{4x_3}(k;\tau)\ ,$$
$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{cum}\{\underbrace{x(n),\cdots x(n)}_{4},\underbrace{x(n+\tau),\cdots x(n+\tau)}_{4}\}e^{-j2\pi(2\alpha_0+\frac{k}{P})n}$$
$$=e^{j8\pi f_e\tau}\widetilde{C}_{8x}(k;\tau)\ ,$$

where $\widetilde{C}_{4x_i}(k;\tau)$, $i=1,2,3$, and $\widetilde{C}_{8x}(k;\tau)$ are defined as in Proposition 2.

Based on (16) and the above equations, the conjugate cyclic spectrum $\widetilde{S}_{2e}(2\alpha_0+k/P;\alpha_0+l/P)$ of Proposition 2 can be established.

References

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